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On the core of a class of location games*

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Abstract. In this paper we introduce a class of cost TU-games arising from continuous single facility location problems. We give some sufficient conditions in order that a game in this class has a non empty core. For the particular subclasses of Weber and minimax location games we study under what conditions the proportionally egalitarian allocation rule selects core allocations.

Zusammenfassung. Ausgehend von kontinuierlichen 1-Standortproblemen wird in diesem Paper eine neue Klasse von kostenbasierten TU-Spielen eingeführt. Es werden einige hinreichende Bedingungen präsentiert unter denen ein Spiel in dieser Klasse einen nicht-leeren Kern hat. Weiterhin werden Zuordnungsregeln für die speziellen Teilklassen der Weber und Minimax Standortspiele vorgeschlagen und es wird untersucht unter welchen Bedingungen die eingeführten Regeln Zuordnungen aus dem Kern wählen.

Key words: Cooperative Games, Core, Location

1 Introduction

One important problem treated by operations research is to find optimal location of facilities, in such a way that the needs of the potential users are satisfied and an objective function, which basically depends on the distances from the users to the facilities, is optimized. This problem gives rise to *location theory*, an operational research branch which has generated a vast literature (for a survey, see Drezner (1995)).

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In the last years, some game theorists have considered operational research problems in which the different elements of the model are controlled by different players, and have treated the question of how are these players going to allocate the benefits (or the costs) if they cooperate and join their forces to implement an optimal solution (from the operations research point of view) of the problem. The precursors of this new treatment can be considered Shapley and Shubik (1972) with their work on assignment games. Two recent surveys on games arising from operations research are Curiel (1997) and Borm et al (2001).

Some cost allocation games arising from location problems have already been described and analyzed. For instance, Granot (1982) studies games associated with single facility location problems in tree graphs, Tamir (1992) considers coverage models on graphs, and Curiel (1997) deals with games arising from p-facilities problems in graphs. However, as far as we know, the cost allocation games associated with continuous location problems have never been approached. In this paper, we define the class of continuous single facility location games and we provide some results on the core of the games in this class. In particular, we obtain two sufficient conditions in order that a continuous single facility location game has a non empty core (section 3), and study under what conditions the proportionally egalitarian allocation rule provides core allocations for Weber and minimax continuous single facility location games (section 4). In section 2 we introduce the classes of games we deal with and motivate the interest of our study.

2 Continuous single facility location games

To start with, we describe what is a continuous single facility location problem. Informally, in such a problem we have a set of n users of a certain facility, placed in n different points in the space $\mathbb{R}^m$ with $m \geq 1$. The problem consists of finding a location for the facility which minimizes the transportation cost (which depends on the distances from the users to the facility). Formally, a continuous single facility location problem is a triplet (N, Φ, d) where:

- $N = \{a_1, \ldots, a_n\}$ is a set of n different points in $\mathbb{R}^m$ (with $n \geq 2$),
- $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous globalizing function satisfying that: 1) Φ is definite, i.e. $\Phi(x) = 0$ if and only if $x = 0$; 2) Φ is monotone, i.e. $\Phi(x) \leq \Phi(y)$ whenever $x \leq y$, and
- $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is a measure of distance, satisfying that, for every $r, s \in \mathbb{R}^m$, $d(r, s) = f(\|r - s\|)$, where f is a lower semicontinuous, non decreasing and non negative map from $\mathbb{R}$ to $\mathbb{R}$ with $f(0) = 0$, and $\| \ \|$ is a norm on $\mathbb{R}^m$.

Solving the continuous single facility location problem (N, Φ, d) for $S \subset N$ means to find an $\bar{x} \in \mathbb{R}^m$ minimizing $\Phi(d^S(x))$, where $d^S(x)$ is the vector in $\mathbb{R}^n$ whose i-th component is equal to $d(x, a_i)$ if $a_i \in S$, and equal to zero otherwise. We denote $L(S) = \min_{x \in \mathbb{R}^m} \Phi(d^S(x))$. It is worth noting that this problem always has a solution for every $S \subset N$ (see, for instance, Plastria (1995)).

This is the classical version of the continuous single facility location problem. Here we consider a natural variant of this problem in which the users in N are interested not only in finding an optimal location of the facility, but also in sharing the corresponding total costs. By total costs we mean the sum of the variable costs (depending on the users and on the location of the facility; they are mostly transportation costs), plus the fixed costs (independent of the number of users and of the location of the facility; they are mostly installation costs). Formally, a continuous single facility location situation is a 4-tuple (N, Φ, d, K) where (N, Φ, d) is a continuous single facility location problem and $K \in \mathbb{R}$, $K \geq 0$, is the fixed installation cost of the facility. Note that we can associate with (N, Φ, d, K) a cost TU-game (N, c) whose characteristic function c is defined, for every $S \subset N = \{a_1, \ldots, a_n\}$, by:

$$c(S) = \begin{cases} K + L(S) & \text{if } S \neq \varnothing \\ 0 & \text{if } S = \varnothing. \end{cases}$$

Every cost TU-game defined in this way is what we call a continuous single facility location game. From now on, for simplicity, we will write location problems, situations or games instead of continuous single facility location problems, situations or games. We denote by $\mathscr{L}(N)$ the class of location games with set of players N (note that we identify the players with their location in the space). As usual, we will also identify the game (N, c) with its characteristic function c.

In a location situation, the goal of the users is to find a location for the facility which minimizes the total cost, and to allocate the corresponding minimal total cost. Before going on, let us give a couple of examples of location situations.

Example 2.1. *Suppose that the councils of n nearby towns (town i located at point $a_i \in \mathbb{R}^2$), make an agreement to build an airport jointly. The building cost of the airport is, approximately, K euros. The agreement includes the compromise to invest in each town a number of euros equal to A times its squared distance to the airport (A being a positive real number) in order to create good roads and railway infrastructure communicating the towns and the airport. The councils want to find an optimal location for the airport (minimizing the total costs) and to share the corresponding total costs. Notice that this is a location situation (N, Φ, d, K) with $\Phi(d^S(x)) = A \sum_{a_i \in S} \|x - a_i\|_2^2$ for all $x \in \mathbb{R}^2$, and all $S \subset N$. Observe that $\Phi(x) = A \sum_{i=1}^n x_i$, for all $x \in \mathbb{R}^n$, $f(y) = y^2$, for all $y \in \mathbb{R}$, and $d = \| \|_2$ is the Euclidean norm. (Here we take $\| \|_2$, but other norms might be more natural in other circumstances.)*

Example 2.2. *Suppose that the councils of n nearby towns (town i located at point $a_i \in \mathbb{R}^2$), make an agreement to create a local TV. This has a fixed cost of K euros (building the main office and studios) and a variable cost. The variable cost (the cost of the station itself) has been estimated to be A times the squared radius of coverage of the TV station (the radius of coverage of a station is the maximum distance to the location of the station from which the TV signal can be properly received; A is a positive real number). The councils want to find an optimal location for the TV station (such that all towns receive the TV signal properly and at a minimum cost) and to share the corresponding total costs. Notice that this is a location situation (N, Φ, d, K) with $\Phi(d^S(x)) = A \max_{a_i \in S} \|x - a_i\|_2^2$ for all $x \in \mathbb{R}^2$, and all $S \subset N$. Observe that $\Phi(x) = A(\max\{x_1, \ldots, x_n\})$, for all $x \in \mathbb{R}^n$, $f(y) = y^2$, for all $y \in \mathbb{R}$, and $d = \| \|_2$ is the Euclidean norm. (Again, we take $\| \|_2$, but other norms might be more natural in other circumstances.)*

An interesting problem which arises now is to study under what conditions there exists a stable allocation of the minimal total costs in a location situation, i.e., under what conditions the core of the corresponding location game is non empty (see section 3 for a formal definition of core). Note that this is an important problem, because users do not only want to find an optimal location for the facility, but also to allocate the total costs. If there is not a stable allocation, probably these users will not be able to reach an agreement and will not build the facility together.

The study of the core of a location game is the main topic of this paper. In the next section we give sufficient conditions in order that the core of a location game is non empty. In section 4 we treat two specially relevant classes of location games and give conditions for the proportionally egalitarian allocation rule (which is commonly used in practice) providing core allocations. It is important to stress that we are looking for conditions that can be checked in a reasonably easy way. Take into account that, in location games, even the computation of the characteristic function can be a difficult task. Hence, we are interested both in finding good theoretical results, and also in producing conditions which are reasonably easy to use in order to predict whether a given location situation is stable.

To conclude this section we study some preliminary properties of location games.

Proposition 2.1. *Take* $c \in \mathscr{L}(N)$ *the location game corresponding to* (N, Φ, d, K). *Then* c *is monotonic (i.e.,* $c(S) \leq c(T)$ *for all* $S, T \subset N$ *with* $S \subset T$*).*

Proof. Let $S \subset T$ be two coalitions. By definition $d_i^S(x) \leq d_i^T(x)$ for all i and x. Then, since Φ is monotone, $\Phi(d^S(x)) \leq \Phi(d^T(x))$. Hence, the result follows. □

Proposition 2.2. *Take* $c \in \mathscr{L}(N)$ *the location game corresponding to* (N, Φ, d, K). *If* $L(N) \leq K$ *then* c *is subadditive (i.e.,* $c(S \cup T)) \leq c(S) + c(T)$ *for all* $S, T \subset N$ *with* $S \cap T = \varnothing$*).*

Proof. Let S, T be two coalitions. Then, by the monotonicity of L (see the proof of Proposition 2.1) and the properties of Φ,

$$L(S \cup T) - (L(S) + L(T)) \leq L(S \cup T) \leq L(N).$$

Now, since $L(N) \leq K$, then $L(S \cup T) \leq K + L(S) + L(T)$ and

$$c(S \cup T) = K + L(S \cup T) \leq K + L(S) + K + L(T) = c(S) + c(T). \quad \square$$

Note that in the result above we proved that, if $L(N) \leq K$, then $c(S \cup T) \leq c(S) + c(T)$ for any pair of coalitions S and T disjoint or not, which is something stronger than the subadditivity of c. One can wonder if a weaker condition can guarantee the subadditivity. In general, this is not true: take, for instance, any two-person location game.

The next example shows a subadditive location game with an empty core. It motivates the next sections of this paper where we search for sufficient conditions for the non emptiness of the core of a location game.

Example 2.3. *Let $N = \{a_1, a_2, a_3\}$ be the set of players, located on the vertices of an equilateral triangle of side l. Consider that the globalizing function is the sum and d is the Euclidean distance to the power of b ($b \geq 2$). Then*

$$\Phi(d^S(x)) = \sum_{a_i \in S} \|x - a_i\|_2^b$$

for every $S \subset N$ and every $x \in \mathbb{R}^m$. It is easy to check that the location game associated with (N, Φ, d, K) is given by:

$$c(a_1) = c(a_2) = c(a_3) = K,$$

$$c(a_1a_2) = c(a_1a_3) = c(a_2a_3) = K + 2(l/2)^b,$$

$$c(a_1a_2a_3) = K + 3\left(\frac{\sqrt{3}}{3}l\right)^b.$$

After some algebra, it can be checked that this game is subadditive if and only if $K \geq (l^b/\sqrt{3}^{b-2}) - (l^b/2^{b-1})$. However, taking for instance $K = (l^b/\sqrt{3}^{b-2}) - (l^b/2^{b-1})$, it can be seen that the resulting location game has an empty core. Namely, since all its players are symmetric, a necessary and sufficient condition for the non emptiness of its core is that the egalitarian allocation $(c(N)/3, c(N)/3, c(N)/3)$ belongs to it. After some algebra it can be checked that this is not the case when $b > 2$.

3 The core

We devote this section to present a sufficient condition for the non emptiness of the core of a location game. Remember that the core of a cost game (N, c) is given by

$$core(c) = \left\{x \in \mathbb{R}^n \middle/ \sum_{i=1}^{n} x_i = c(N), c(S) \geq \sum_{i \in S} x_i \ \forall S \subset N\right\}.$$

First, we prove a technical result concerning the sum of the balancing coefficients of a balanced family of coalitions. Recall that a collection of coalitions $\mathscr{B} \subset 2^N$ is balanced if and only if there exists a set of positive real coefficients $\{\gamma_S / S \in \mathscr{B}\}$ (balancing coefficients) satisfying that $\sum_{S:a_i \in S} \gamma_S = 1$ for every $a_i \in N$. The set of balancing coefficients associated with a balanced collection needs not to be unique. However, every minimal balanced collection of coalitions (in the sense that it does not properly contain another balanced collection) has a unique set of balancing coefficients (see Owen (1995)). It is a well-known result that a cost game (N, c) has a non empty core if and only if it, for every minimal balanced collection $\mathscr{B}$ with balancing coefficients $\{\gamma_S / S \in \mathscr{B}\}$, it holds that $\sum_{S \in \mathscr{B}} \gamma_S c(S) \geq c(N)$ (again, see Owen (1995)).

Note that the only balanced collection with balancing coefficients summing up to one is $\mathscr{B} = \{N\}$. Indeed, for every balanced collection $\mathscr{B}$ and every $a_i \in N$, $\sum_{S:a_i \in S} \gamma_S = 1$; if, in addition, $\sum_{S \in \mathscr{B}} \gamma_S = 1$ then, for every $S \in \mathscr{B}$ and every

$a_i \in N$, $a_i \in S$. Therefore $S = N$, for all $S \in \mathscr{B}$, and thus $\mathscr{B} = \{N\}$. We say that $\mathscr{B} = \{N\}$ is the trivial collection. Our next result establishes bounds on the sum of the balancing coefficients for any non trivial balanced collection.

Lemma 3.1. *Let $\mathscr{B}$ be a non trivial balanced collection with balancing coefficients $\{\gamma_S / S \in \mathscr{B}\}$. Then,*

$$\frac{n}{n-1} \leq \sum_{S \in \mathscr{B}} \gamma_S \leq n.$$

Proof. Let us consider the following linear programming problem (1):

$$\begin{aligned} \min \quad & \sum_{S \in 2^N \setminus \{N\}} \gamma_S \\ s.t.: \quad & \sum_{\{S \in 2^N \setminus \{N\} : a_i \in S\}} \gamma_S = 1 \quad \forall a_i \in N \\ & \gamma_S \geq 0 \quad \forall S \in 2^N \setminus \{N\}. \end{aligned} \tag{1}$$

A solution to this problem is a set of balancing coefficients of a non trivial balancing collection with a minimal sum ($\mathscr{B} = \{S \in 2^N / \gamma_S > 0\}$). Let us denote the coalition $N \setminus \{a_j\} = \{a_1, a_2, .., a_{j-1}, a_{j+1}, \ldots, a_n\}$ by $-j$. Consider the basis B of Problem (1) of the columns which correspond to $\gamma_{-1}, \gamma_{-2}, \ldots, \gamma_{-n}$. In this problem the matrix of B, its inverse B^{-1} and the transformed right-hand side $B^{-1}b$ are:

$$B = \begin{bmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{bmatrix},$$

$$B^{-1} = \frac{1}{n-1} \begin{bmatrix} -(n-2) & 1 & \ldots & 1 \\ 1 & -(n-2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & -(n-2) \end{bmatrix},$$

and $B^{-1}b = \left[\frac{1}{n-1}, \ldots, \frac{1}{n-1}\right]^t$. The reduced costs for any coalition S with $1 \leq k \leq n-1$ players are:

$$c_B B^{-1} a_S - c_S = \frac{k}{n-1} - 1 < 0 \quad \text{iff } k < n-1,$$

$$c_B B^{-1} a_S - c_S = \frac{n-1}{n-1} - 1 = 0 \quad \text{iff } k = n-1.$$

Then B is a basis associated with an optimal solution of Problem (1), which proves the lower bound. The proof for the upper bound is straightforward and it follows taking the collection whose elements are all the sets of size one with coefficients equal to 1. □

Using the lemma above, we prove now the main result in this section.

Theorem 3.1. *Let (N, Φ, d, K) be a location situation and let (N, c) be its corresponding location game. Denote $l_2 = \min_{S \subset N:|S|=2} L(S)$.*

a) *Suppose that $2 \le n \le 2 + \frac{l_2}{K}$. If $K(n-1) \ge L(N)$, then c has a non empty core.*
b) *Suppose that $2 + \frac{l_2}{K} < n$. If $K \ge (n-1)L(N) - nl_2$, then c has a non empty core.*

Proof. In a location game we have for any balanced collection $\mathscr{B}$ with balancing coefficients $\{\gamma_S / S \in \mathscr{B}\}$:

$$\sum_{S \in \mathscr{B}} \gamma_S c(S) = K\left(\sum_{S \in \mathscr{B}} \gamma_S\right) + \sum_{S \in \mathscr{B}} \gamma_S L(S).$$

Taking into account the monotonicity of L and the fact that $L(S) = 0$ for any coalition S of size one, we have that

$$\begin{aligned}\sum_{S \in \mathscr{B}} \gamma_S c(S) &= K\left(\sum_{S \in \mathscr{B}} \gamma_S\right) + \sum_{S \in \mathscr{B}:|S| \ge 2} \gamma_S L(S) \\ &\ge K\left(\sum_{S \in \mathscr{B}} \gamma_S\right) + \sum_{S \in \mathscr{B}:|S| \ge 2} \gamma_S l_2.\end{aligned}$$

For every minimal balanced collection $\mathscr{B}$ denote

$$m(\mathscr{B}) = K\left(\sum_{S \in \mathscr{B}} \gamma_S\right) + l_2 \sum_{S \in \mathscr{B}:|S| \ge 2} \gamma_S$$

(note that, if $\mathscr{B}$ is minimal, the balancing coefficients are uniquely determined). Then, a sufficient condition for the non emptiness of the core is that

$$\min_{\{\mathscr{B}:\mathscr{B} \text{ non trivial and minimal balanced}\}} m(\mathscr{B}) \ge c(N). \tag{2}$$

Suppose that this minimum is achieved in $\hat{\mathscr{B}}$. If $\{a_i\} \notin \hat{\mathscr{B}}$ for every $a_i \in N$, then $\hat{\mathscr{B}} = \{-i / a_i \in N\}$ (see Lemma 3.1) and $m(\hat{\mathscr{B}}) = (K + l_2)\frac{n}{n-1}$. If $\hat{\mathscr{B}} = \{\{a_i\} / a_i \in N\}$, then $m(\hat{\mathscr{B}}) = Kn$. In any other case $\hat{\mathscr{B}}$ can only be a family $\{\{a_i\}, N \backslash a_i\}$ (for an $a_i \in A$) and, then, $m(\hat{\mathscr{B}}) = 2K + l_2$.

Now, since $m(\hat{\mathscr{B}}) = \min\left\{(K + l_2)\frac{n}{n-1}, Kn, 2K + l_2\right\}$, then it can be easily checked that

$$m(\hat{\mathscr{B}}) = \begin{cases} Kn & \text{if } 2 \le n \le 2 + \dfrac{l_2}{K} \\ (K + l_2)\dfrac{n}{n-1} & \text{if } 2 + \dfrac{l_2}{K} < n. \end{cases}$$

This together with (2) completes the proof. □

The following examples show that the bounds in the theorem are tight, in the sense that they cannot be improved for all n. In particular, these examples show that they are achieved for $n = 2$ and $n = 3$.

Example 3.1. *Let $N = \{a_1, a_2\}$ be the set of players, located on the extremes of a segment of length 2. Consider that the globalizing function is the sum and that d is the squared Euclidean distance. It is easy to check that the location game (N, c) associated with (N, Φ, d, K) is given by: $c(a_1) = c(a_2) = K$, $c(a_1a_2) = K + 2$. Clearly, this game has a non empty core if and only if $K \ge 2$. Note that, in this case, we are under condition a) and $K(n-1) \ge L(N)$ is equivalent to $K \ge 2$, so the bound is tight for this game.*

Example 3.2. *Take the location situation and the location game of Example 2.3 with $b = 2$. Thus, $N = \{a_1, a_2, a_3\}$ and the characteristic function of the game is:*

$$c(a_1) = c(a_2) = c(a_3) = K$$

$$c(a_1a_2) = c(a_2a_3) = c(a_1a_3) = K + \frac{l^2}{2}$$

$$c(a_1a_2a_3) = K + l^2.$$

Since players are symmetric in c, $core(c) \neq \emptyset$ if and only if the egalitarian allocation $\left(\frac{c(N)}{3}, \frac{c(N)}{3}, \frac{c(N)}{3}\right)$ belongs to $core(c)$. It is easy to check that this allocation belongs to the core if and only if $K \ge \frac{l^2}{2}$. Note that, in this case, if $K > \frac{l^2}{2}$ we are under condition b); if $K \le \frac{l^2}{2}$ we are under condition a). In both cases, the bound given by the theorem is $K \ge \frac{l^2}{2}$. So, again, the bound is tight in this example.

We have given a sufficient condition for the non emptiness of the core of any location game. This condition is a good one because: a) it cannot be improved in general, and b) it can be checked in a reasonably easy way (you only have to compute l_2 and $L(N)$). Another sufficient condition for the non emptiness of the core which is even simpler is given below.

Proposition 3.1. *Let (N, Φ, d, K) be a location situation and let (N, c) be its corresponding location game. If $K \ge (n-1)L(N)$, then c has a non empty core.*

Proof. Let us check that the egalitarian allocation $\left(\frac{K+L(N)}{n},\ldots,\frac{K+L(N)}{n}\right)$ belongs to $core(c)$. Namely, for every $S \subset N$ with $|S| \leq n-1$,

$$|S|\frac{K+L(N)}{n} \leq (n-1)\frac{K+L(N)}{n} \leq K \leq c(S). \quad \square$$

Note that, although the condition in Proposition 3.1 is simpler than the condition in Theorem 3.1, it is also weaker. Only in case l_2 is very small (i.e., in case there are two users located in two points very close), the condition in Theorem 3.1 tends to be the same as that of Proposition 3.1. But, in such a case, perhaps it would be more convenient to consider these two close players as only one player.

4 Two special classes of location games

In this section we deal with two classes of location games corresponding to two important classes of location problems: Weber location problems and minimax location problems. In both classes d is the squared Euclidean distance. In the Weber location problem the globalizing function is the sum, whereas in the minimax location problem the globalizing function is the maximum. Hence, the corresponding location games are given by

$$c_W(S) = \min_{x \in \mathbb{R}^m} \sum_{a \in S} \|x-a\|_2^2 + K \tag{3}$$

$$c_M(S) = \min_{x \in \mathbb{R}^m} \max_{a \in S} \|x-a\|_2^2 + K \tag{4}$$

for all non empty $S \subset N$ (being (N, Φ, d, K) a Weber and a minimax location situation, respectively). Note that Example 2.1 and Example 2.2 in section 2 are a Weber location situation and a minimax location situation, respectively.

These two classes of location problems are well-known (see Love et al (1988)) and have been extensively studied in the literature of location theory. It is straightforward to derive that the optimal solution of Problem (3) is:

$$x^*(S) = \frac{1}{|S|}\sum_{a \in S} a.$$

On the other hand, the optimal solutions of problem (4) can be easily obtained in $O(nlogn)$ time (see Megiddo (1983) or Megiddo and Zemel (1986)).

In this section we deal with the location games arising from Weber and minimax location situations. Obviously, all the results of the previous section still apply for these games. However, now we go further and, instead of wondering when their cores are non empty, we study under which conditions a very natural solution concept (the proportionally egalitarian solution) provides core allocations for Weber or minimax location games.

In many practical situations, when several users decide to build a facility together, they agree to use some sort of proportionally egalitarian solution for allocating the costs. For Weber location situations a proportionally egalitarian rule consists of the following: a) the facility will be built in the location $x^*(N)$

which minimizes the total cost, b) the fixed costs K are proportionally divided among the users, according to a vector of positive real proportionality coefficients $(\alpha_1, \dots, \alpha_n)$, and c) each user pays the transportation costs he produces (provided that the facility will be located in $x^*(N)$).

This means that, if (N, c_W) is the Weber location game associated with the Weber location situation (N, Φ, d, K), according to this rule (that we denote by E), user i must pay

$$E_i(c_W) = K_i + \|x^*(N) - a_i\|_2^2,$$

where $K_i = K\dfrac{\alpha_i}{\alpha}$, and $\alpha = \sum_{i=1}^n \alpha_i$. For a better understanding of item b), let us say that, in many cases, the proportionality vector will be $(1/n, \dots, 1/n)$, but sometimes another vector will be more appropriate. For instance, in Example 2.1, the towns can use their number of inhabitants as proportionality coefficients.

Now we give a necessary and sufficient condition in order that this proportional egalitarian solution provides core allocations.

Theorem 4.1. *Let* (N, c_W) *be a Weber location game corresponding to the Weber location situation* (N, Φ, d, K). *Then* $E(c_W)$ *belongs to core*(c_W) *if and only if*

$$|S|\,\|x^*(S) - x^*(N)\|_2^2 \le \sum_{a_i \notin S} K_i \tag{5}$$

for any coalition $S \subset N$.

Proof. $E(c_W)$ belongs to *core*(c) if and only if, for all $S \subset N$,

$$\sum_{a_i \in S} K_i + \sum_{a_i \in S} \|x^*(N) - a_i\|_2^2 \le K + \sum_{a_i \in S} \|x^*(S) - a_i\|_2^2$$

or, equivalently,

$$\sum_{a_i \in S} \|x^*(N) - a_i\|_2^2 - \sum_{a_i \in S} \|x^*(S) - a_i\|_2^2 \le \sum_{a_i \notin S} K_i.$$

But

$$\begin{aligned}
&\sum_{a_i \in S} (\|x^*(N) - a_i\|_2^2 - \|x^*(S) - a_i\|_2^2) \\
&\quad = \sum_{a_i \in S} [\|x^*(N)\|_2^2 - \|x^*(S)\|_2^2 - 2\langle x^*(N) - x^*(S), a_i\rangle] \\
&\quad = |S| \left[\|x^*(N)\|_2^2 - \|x^*(S)\|_2^2 - 2\sum_{a_i \in S} \left\langle x^*(N) - x^*(S), \frac{a_i}{|S|} \right\rangle \right] \\
&\quad = |S|[\|x^*(N)\|_2^2 + \|x^*(S)\|_2^2 - 2\langle x^*(N), x^*(S)\rangle].
\end{aligned}$$

Hence, the result follows. $\square$

Condition (5) can be interpreted in the following way. The left hand side of the inequality can be roughly seen as the extra transportation cost that users in S will have to hold if the grand coalition N forms. The right hand side is the part of K supported by the players in $N \backslash S$. Hence, $E(c_W)$ belongs to $core(c_W)$ if and only if the extra transportation cost that every coalition must support is smaller than or equal to the fixed cost that it saves. It is worth noting that condition (5) is fulfilled, for every Weber location problem and every proportionality vector α, if K is large enough.

An easier condition which does not depend on the set of coalitions S is derived in our next result. Set $r_N = \max_{a \in N} \|x^*(N) - a\|_2^2$.

Corollary 4.1. *$E(c_W) \in core(c)$ if*

$$(n-1) r_N \leq \min_{a_i \in N} K_i.$$

Proof. Note first that $r_N \geq \|x^*(N) - x^*(S)\|_2^2$ for any coalition $S \subset N$. Then, under the hypothesis of the corollary, $r_N \leq \frac{\min_{a_i \in N} K_i}{(n-1)} \leq \frac{\sum_{a_i \notin S} K_i}{n-1} \leq \frac{\sum_{a_i \notin S} K_i}{|S|}$ for any coalition $S \subset N$, $S \neq N$. Thus, $|S| \, \|x^*(N) - x^*(S)\|_2^2 \leq \sum_{a_i \notin S} K_i$, and we obtain from (5) that $x_P \in core(c)$ (note that (5) is obviously true for $S = N$). □

Let us consider now the class of minimax location games in $\mathbb{R}^m$. To start with note that, in this class,

$$l_2 = \min_{a, b \in N, a \neq b} \frac{\|a - b\|_2^2}{4}.$$

Hence, l_2 can be computed in a reasonably easy way and the conditions in Theorem 3.1 can be easily checked.

Now denote $l_k = \min_{S \subset N : |S| = k} L(S)$, for $k \in \{2, \ldots, n\}$. In location theory, it is a well-known feature that, for every $S \subset N$ with $|S| \geq m+1$, $L(S)$ is equal to $L(\bar{S})$ for an $\bar{S} \subset S$ with $|\bar{S}| = m+1$ (because, in a minimax location problem with set of points S, the solution is the center of the smallest sphere containing S, and this sphere is fully determined by at most $m+1$ points in S, only three points for a circle in $\mathbb{R}^2$, see Elzinga and Hearn (1972)). Hence, since L is monotone, $l_k = l_{m+1}$ for all $k \geq m+1$.

Next we provide a sufficient and easy to check condition in order that the proportionally egalitarian solution belongs to the core in a minimax location game. First we say what we mean by proportionally egalitarian solution in this context. As in Weber location games, assume that there is a vector of positive real proportionality coefficients $(\alpha_1, \ldots, \alpha_n)$. Assume, without loss of generality, that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Denote $\alpha = \sum_{a_i \in N} \alpha_i$. Note that, in minimax location games, a particular user does not produce transportation costs once the location of the facility is decided, in the sense that the transportation costs would be the same even if he leaves the game. (Think of Example 2.2: once the location and the radius of coverage of the station have been decided, a particular user does not produce transportation costs. This is not the case in a Weber location game, as the one in Example 2.1). Hence, the right way of defining here the proportionally egalitarian solution E is the following. If (N, c_M) is the minimax location game associated with the minimax location situation (N, Φ, d, K), then $E_i(c_M) = (K + L(N))\alpha_i/\alpha$, for all $a_i \in N$. The next theorem states a sufficient condition for E providing core allocations.

Theorem 4.2. *Let (N, c_M) be the minimax location game corresponding to the minimax location situation (N, Φ, d, K). Then, $E(c_M)$ belongs to core(c_M) if*

$$L(N)\frac{\sum_{j=n-k+1}^{n}\alpha_j}{\alpha} - r(k) \leq K\left(1 - \frac{\sum_{j=n-k+1}^{n}\alpha_j}{\alpha}\right) \tag{6}$$

for every $k \in \{1, \ldots, n\}$, where $r(1) = 0$, $r(k) = l_k$ for any $2 \leq k \leq m+1$ and $r(k) = l_{m+1}$ for all $k \geq m+1$.

Proof. $E(c_M) \in core(c_M)$ if and only if, for all $S \subset N$,

$$\sum_{a_i \in S}(K + L(N))\frac{\alpha_i}{\alpha} \leq K + L(S). \tag{7}$$

Taking into account that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and that $l_k = l_{m+1}$ for all $k \geq m+1$, it is clear that, for every $k \in \{1, \ldots, n\}$, (6) implies (7) for every $S \subset N$ with $|S| = k$. □

Condition (6) can be interpreted in the following way. The left hand side of the inequality is the part of the coverage cost that the players in $\{a_1, \ldots, a_k\}$ save, minus the smallest coverage cost which should be paid by a coalition of k players if only they cooperate. The right hand side is the part of the fixed cost payed by players in $\{a_1, \ldots, a_k\}$. Hence, $E(c_M)$ belongs to *core*(c_M) if and only if the players with a small proportionality coefficient do pay a large enough part of the coverage cost. Again, it is worth noting that condition (6) is fulfilled, for every minimax location problem and every proportionality vector α, if K is large enough.

Note that a weaker sufficient condition could be easily found. However, that of Theorem 4.2 above is specially easy to check. You have only to consider n inequalities, and to compute $l_2, \ldots l_{m+1}$ which in the case of the plane $(\mathbb{R}^2)$ reduces to l_2 and l_3.

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